THE EFFECT OF EXTERNAL PERTURBING MOMENTS ON THE DYNAMICS OF A UNIAXIAL SINGLE-FLYWHEEL ATTITUDE CONTROL SYSTEM OF A SPACECRAFT

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Effective damping of relative craft oscillations in a single-flywheel uniaxial attitude control system whose purpose is to keep a spacecraft oriented in some direction in space, e.g. towards the Sun, can be achieved by nonrigid coupling of its body with the axis of rotation of the flywheel [1]. This can be accomplished by adequately restricting two of the degrees of freedom of the flywheel in a three-degree universal suspension by means of a proportional damper of any type.

In the present paper we shall consider the stability of free oscillations of a spacecraft equipped with a flywheel on a movable axis, as well as the motion of the oriented craft axis under the action of external perturbing moments and the perturbed motion of the craft itself relative to the oriented axis.

1. Basic equations. Unperturbed motion. Let the frame of the spacecraft be connected with the system of axes $Cxyz$ which we consider to be the central system for the frame and for all the attached movable bodies considered as point masses. The latter include the flywheel with its suspension elements. We assume that it is the z-axis of the spacecraft which must be directed towards the Sun.

Let us begin with the simplest formulation of the problem in which we assume that the oscillations of the z-axis of the craft relative to the solar direction and of the axis of rotation of the flywheel relative to the craft body are small. If the axes $C_1 x_1 y_1 z_1$ (where the axis z_1 is directed along the axis of rotation of the flywheel which experiences small oscillations in the axes xyz), then the orientation $C_1 x_1 y_1 z_1$ relative to the axes $C_1 xyz$ can be readily defined in terms of the Krylov angles α_1, β_1, φ_1.

Applying precession theory, i.e. neglecting the equatorial components of the kinetic moment of the flywheel as well as the kinetic moments of its suspension elements, we obtain the simplest equations of rotational motion of the spacecraft in the form

$$\begin{aligned}
&A\frac{d\omega_x}{dt} + (C-B)\,\omega_y\omega_z + H\left(\frac{d\beta_1}{dt} + \omega_y - \omega_z\alpha_1\right) = M_x \\
&B\frac{d\omega_y}{dt} + (A-C)\,\omega_x\omega_z + H\left(\frac{d\alpha_1}{dt} - \omega_x + \omega_z\beta_1\right) = M_y \qquad (1.1)\\
&C\frac{d\omega_z}{dt} + (B-A)\,\omega_x\omega_y + \frac{dH}{dt} = M_z \qquad (H = I\varphi_1')
\end{aligned}$$

Here A, B, and C are the principal central moments of inertia of the craft with the attached point masses; ω_x, ω_y, and ω_z are the projections of the absolute angular velocity vector of the craft; M_x, M_y, and M_z are the projections of the vector of the principal moment of the external forces relative to the point C; H is the characteristic kinetic moment of the flywheel.

Henceforth we shall confine our attention to the case of steadystate motion of the flywheel (H = const).

The equations of precessional motion of the flywheel rotor under the above assumptions can be reduced to

$$-H\,(d\beta_1/dt+\omega_y-\omega_z\alpha_1)=Q_1,\quad H\,(d\alpha_1/dt-\omega_x+\omega_z\beta_1)=Q_2 \tag{1.2}$$

where the generalized forces Q_1 and Q_2 are determined largely by the elastic and damping moments of the shock absorbers of the flywheel suspension and by other resistance moments acting along the corresponding axes of rotation of the universal suspension. If the flywheel suspension is restricted by the shock absorbers according to a linear law, we can assume that

$$Q_1\approx -h\,d\alpha_1\,/\,dt-k\alpha_1,\qquad Q_2\approx -h\,d\beta_1\,/\,dt-k\beta_1 \tag{1.3}$$

where h and k are the damping and elastic constants on the flywheel suspension axes.

If we introduce some system of the solar tracking axes $Cx_0y_0z_0$ and define the orientation of the spacecraft axes $Cxyz$ in this system in terms of the same Krylov angles α, β, and φ, then the expressions for the projections of the absolute angular velocity of the spacecraft on its axes xyz in (1.1) and (1.2) become

$$\omega_x=-\frac{d\alpha}{dt}\cos\varphi+\left(\frac{d\beta}{dt}+n_0\right)\cos\alpha\sin\varphi \tag{1.4}$$

$$\omega_y=\frac{d\alpha}{dt}\sin\varphi+\left(\frac{d\beta}{dt}+n_0\right)\cos\alpha\cos\varphi,\qquad \omega_z=\frac{d\varphi}{dt}+\left(\frac{d\beta}{dt}+n_0\right)\sin\alpha$$

where the small quantity $n_0\approx 0.986^\circ$/day is the angular velocity of the Earth's motion around the Sun (we assume that the spacecraft moves in an orbit around the Earth).

Henceforth we shall assume that the spacecraft is dynamically symmetrical, i.e. that $A=B$. Let us introduce the following complex variables and dimensionless parameters into our investigation:

$$\omega=\omega_x+i\omega_y,\ \gamma=-\alpha_1+i\beta_1,\ \varepsilon=\frac{C}{A},\ k_1=\frac{kA}{H^2},\ h_1=\frac{h}{H},\ \omega_0=\frac{A\omega_z}{H}(i=\sqrt{-1}) \tag{1.5}$$

Eqs. (1.1) to (1.3) then become

$$\frac{A}{H}\frac{d\omega}{dt}+i\,[(1-\varepsilon)\,\omega_0-1]\,\omega+\omega_z\gamma-i\,\frac{d\gamma}{dt}=\frac{1}{H}(M_x+iM_y)$$

$$\frac{A}{H}(1+ih_1)\frac{d\gamma}{dt}+i\,(\omega_0+k_1)\,\gamma+\frac{A}{H}\,\omega=0,\ C=\frac{d\omega_z}{dt}=M_z \tag{1.6}$$

For unperturbed motion in the absence of external moments we have

$$\omega_z=\text{const} \tag{1.7}$$

and system (1.6) assumes its simplest form, becoming a system of linear differential equations with constant and complex coefficients. The necessary and sufficient conditions of stability of the solutions of system (1.6) with complex coefficients in the presence of damping in the flywheel suspension are the inequalities [2]

$$1+k_1+\omega_0>0,\qquad (1+\varepsilon\omega_0)\,\{k_1[1-(1-\varepsilon)\,\omega_0]-\omega_0^2\,(1-\varepsilon)\}>0 \tag{1.8}$$

For unperturbed motion, conditions (1.8) serve to interrelate the inertial parameters of the craft ($\varepsilon\lesseqgtr 1$), the value of the kinetic moment of the flywheel ($H>0$), the rigidity parameter of the restriction in its suspension, and the rotational velocity of the body about the oriented axis in such a way as to permit stable damping of the craft oscillations about the direction of orientation. In dimensional parameters (1.5) the stability domains are given by the inequalities

$$\omega_0>-\frac{1}{\varepsilon},\qquad \omega_0>-(k_1+1),\qquad k_1>\frac{(1-\varepsilon)\,\omega_0^2}{1-(1-\varepsilon)\,\omega_0} \tag{1.9}$$

Upon fulfillment of conditions (1.9) the roots z_i of the characteristic equation of system

(1.6) (which is a second-order system with complex coefficients) can be determined from Formulas

$$
\begin{aligned}
z_1 &= \frac{H}{A}(\lambda_1 + i\omega_1), \qquad z_2 = \frac{H}{A}(\lambda_2 + i\omega_2) \qquad \text{for } f_2 > 0 \\
z_1 &= \frac{H}{A}(\lambda_1 + i\omega_2), \qquad z_2 = \frac{H}{A}(\lambda_2 + i\omega_1) \qquad \text{for } f_2 < 0
\end{aligned} \tag{1.10}
$$

in which we use the notation

$$
\begin{gathered}
\lambda_{1,2} = -\frac{h_1 d_2}{2(1+h_1^2)} \pm \frac{1}{\sqrt{2}}[(f_1^2 + f_2^2)^{1/2} + f_1]^{1/2}, \quad \omega_{1,2} = -\frac{1}{2}\left[\frac{d_2}{1+h_1^2} - d_1\right] \pm \\
\pm \frac{1}{\sqrt{2}}[(f_1^2 + f_2^2)^{1/2} - f_1]^{1/2} \\
d_1 = 1 - (1-\varepsilon)\omega_0, \qquad d_2 = 1 + k_1 + \omega_0 \\
f_1 = \frac{1}{4}\left[\left(\frac{h_1 d_2}{1+h_1^2}\right)^2 - \left(\frac{d_2}{1+h_1^2} - d_1\right)^2\right] - \frac{1}{1+h_1^2}[k_1 d_1 - (1-\varepsilon)\omega_0^2] \\
f_2 = \frac{h_1}{1+h_1^2}\left[\frac{d_2}{2}\left(\frac{d_2}{1+h_1^2} - d_1\right) + k_1 d_1 - (1-\varepsilon)\omega_0^2\right]
\end{gathered} \tag{1.11}
$$

The roots of initial fourth-order system (1.1) to (1.3) are determined by the values of z_i and of the conjugates $\bar{z}_i$ $(i = 1, 2)$. Since $d_2 > 0$ by virtue of stability conditions (1.9), and since $[(f_1^2 + f_2^2)^{1/2} + f_1]^{1/2} > 0$, the real part of the roots of minimal absolute value is determined by the value of λ_1. This quantity can be optimized by suitable choice of the system parameters, which in turn guarantees the most rapid damping out of the oscillations of the flywheel axis and of the oriented axis of the craft. On termination of the transient process in unperturbed motion the longitudinal axis of the craft does not coincide exactly with the direction of the Sun, but rather with the direction of the moment-of-momentum vector of the system (we neglect the slow motion of the Earth relative to the Sun) which differs from the solar direction by the angle

$$
\zeta_\infty = \frac{1}{C\varphi^\cdot + H}[(C\varphi^\cdot + H)\zeta_0 + H\eta_0 + iA\zeta_0^\cdot]
$$

(see Notation (2.1)). The craft rotates about its longitudinal axis with the velocity $d\varphi/dt =$ $=$ const.

2. Perturbed motion of the oriented axis. In perturbed motion the right sides of Eqs. (1.1) contain the moments of gravitational, aerodynamic, magnetic, and other forces. Introducing the complex coordinates

$$
\gamma = \eta e^{-i\varphi}, \qquad \zeta = -\alpha + i\beta, \qquad \omega = (d\zeta/dt + in_0)e^{-i\varphi} \tag{2.1}
$$

and making use of the matrix $\| \beta_{is}^\cdot \|$ of direction cosines between the axes xyz of the craft and the solar tracking axes x_0, y_0, z_0, we can rewrite Eqs. (1.6) of craft and its orientation system motion as

$$
\begin{gathered}
\frac{A}{H}\frac{d^2\zeta}{dt^2} - i(1+\varepsilon\omega_0)\frac{d\zeta}{dt} - i\frac{d\eta}{dt} = \frac{1}{H}(M_x + iM_y)e^{i\varphi} - n_0(1+\varepsilon\omega_0) \\
C\frac{d^2\varphi}{dt^2} = M_z, \qquad \frac{A}{H}\frac{d\zeta}{dt} + \frac{A}{H}(1+ih_1)\frac{d\eta}{dt} + (h_1\omega_0 + ik_1)\eta = -i\frac{A}{H}n_0
\end{gathered} \tag{2.2}
$$

Since we are considering an oriented object, the expressions for the projections of the moment of external forces in (2.2) can be represented as a sum of two moments. The first of these depends on the motion of the coordinate system in which the object must be oriented and on the angle φ of characteristic rotation of the craft the second moment depends on this motion and on the orientation of the craft relative to the chosen coordinate system. We can illustrate this for the gravitational force moment. For a dynamically symmetrical body

$$M_x \approx 3\,\frac{\mu}{r^3}\,(C - A)\,\delta_2\delta_3, \qquad M_g \approx 3\,\frac{M}{r^3}\,(A - C)\,\delta_1\delta_3, \qquad M_z = 0 \tag{2.3}$$

where μ is the gravitational constant, r is the distance between the attracting center and the point C, and δ_i are the direction cosines of r relative to the craft axes $Cxyz$ which are given by

$$\delta_i = \mu_1\beta_{i1} + \mu_2\beta_{i2} + \mu_3\beta_{i3} \qquad (\mu_i = a_i \cos u + b_i \sin u) \qquad (i = 1, 2, 3) \tag{2.4}$$

Here u is the angle of true anomaly of the craft and a_i, b_i, and c_i are the direction cosines between the solar tracking axes and the system of axes x^*, y^*, z^* related to the orbit in such a way that x^* and y^* lie in the orbit plane (x^* being directed along the line of nodes, y^* along the direction of craft motion, and z^* perpendicularly to the orbit plane); a_i, b_i, and c_i are slowly varying functions which depend on the orientation of the orbit relative to the solar tracking axes. After some uncomplicated operations, the expression for the gravitational force moments in (2.2) becomes (2.5)

$$(M_x + iM_y)\,e^{i\varphi} = -3A\,(1-\varepsilon)\,\omega_*^{\,2}\,\{\mu_3(\mu_2 - i\mu_1) + [1 - {}^3/_2(\mu_1^2 + \mu_2^2)]^2\zeta + {}^1/_2(\mu_1 + i\mu_2)^2\bar{\zeta}\}$$

$$(\omega_* = \mu / r^3)$$

From now on we shall assume that the craft is in circular orbit (i.e. that $\omega_* =$ const).

In the general case where the object is characterized by some asymmetry (dynamic, magnetic, or aerodynamic) the general expressions for the projections of the moment of external forces for motion of the craft along an arbitrary orbit become

$$(M_x + iM_y)\,e^{i\varphi} = m \sum_{k=0}^{N} (P_k + iR_k)\,e^{ik\varphi}, \qquad M_z = m_z \sum_{k=1}^{N} (A_k \cos k\varphi + B_k \sin k\varphi) \tag{2.6}$$

to within terms of the zeroth order of smallness.

Here the parameters m and m_z characterize the magnitudes of the moment of external forces; P_k, R_k, A_k, and B_k are periodic functions of the latitude argument and can be expressed as finite sums of the sines and cosines of this argument,

$$P_k = p_{0k} + \sum_{\nu=1}^{M} (p_{\nu k} \cos \nu u + q_{\nu k} \sin \nu u), \qquad R_k = r_{0k} + \sum_{\nu=1}^{M} (r_{\nu k} \cos \nu u + s_{\nu k} \sin \nu u) \tag{2.7}$$

$$A_k = a_{0k} + \sum_{\nu=1}^{M} (a_{\nu k} \cos \nu u + c_{\nu k} \sin \nu u), \qquad B_k = b_{0k} + \sum_{\nu=1}^{M} (b_{\nu k} \cos \nu u + d_{\nu k} \sin \nu u)$$

which are valid in practically all cases.

For example, the moment expressions for the gravitational force moment acting on a dynamically asymmetrical object ($A \neq B \neq C$) become

$$(M_x + iM_y)\,e^{i\varphi} = 3\omega_*^2\,[C - {}^1/_2(A + B)]\,\mu_3\,(\mu_2 - i\mu_1) - {}^3/_2\,(B - A)\;\omega_*^2\mu_3\;(\mu_2 + i\mu_1)e^{2i\varphi}$$

$$M_z = 3\,(B - A)\,\omega_*^2\,[{}^1/_2\,(\mu_2^2 - \mu_1^2)\sin 2\varphi + \mu_1\mu_2\cos 2\varphi] \tag{2.8}$$

The upper limits of the sums in the right sides of Formulas (2.6) and (2.7) can practically always be assumed to be positive integers. Their values are determined by the degree of accuracy with which the external force moment is computed. The numbers N and M are usually small. Thus, we find from the above formulas that for the gravitational force moment $N = 2$ and $M = 2$; for the magnetic force moment under the usual assumptions we have $N = 2$ and $M = 4$. The case of the aerodynamic moment is more complex, since here the sums in the right sides of (2.6) and (2.7) are sums of several of the initial terms of a Fourier series Our computations for a specific spacecraft geometry showed, however that even here it is sufficient to take only a very small number of harmonics in the expansion.

We can estimate the effect of terms of zero order of smallness in the expression for the external force moment on the deviation of the z-axis of the craft from the solar direction under the condition that $d\varphi/dt =$ const. This effect is determined by the following particular solution of system (2.2):

$$\zeta_* = \left[-in_0 + \frac{A}{H}\frac{(-r_{00}+ip_{00})}{1+\varepsilon\omega_0}\right]t \tag{2.9}$$

The above solution is associated with the zeroth harmonic ($k = 0$) of expansion (2.7), i.e. with the moment of forces acting on an ideally symmetrical craft. In particular, in the presence of the gravitational force moment alone

$$\zeta_* = i\left[-n_0 + \frac{3A(1-\varepsilon)\omega_*^2}{H(1+\varepsilon\omega_0)}c_3(c_2 - ic_1)\right]t \tag{2.10}$$

Rewriting expansion (2.7) in the form

$$(M_x + iM_y)e^{i\omega} = m\sum_{k=0}^{N}(p_{0k} + ir_{0k})e^{ik\varphi} + \tag{2.11}$$

$$+\frac{m}{2}\sum_{k=0}^{N}\sum_{\nu=1}^{M}\{[(p_{\nu k}+s_{\nu k}) + i(r_{\nu k}-q_{\nu k})]e^{i(\nu u+k\varphi)} + [(p_{\nu k}-s_{\nu k}) + i(r_{\nu k}+q_{\nu k})]e^{i(-\nu u+k\varphi)}\}$$

we note that in the resonance case for $k\dot{\varphi} = \pm\nu\dot{u}$ expansion (2.11) contains not only the zeroth harmonic, but also an additional constant which alters solution (2.9) quantitatively only.

Consideration of the effect of the components of the external moments dependent of the orientation of the craft entails investigation of differential equations with variable coefficients. With allowance for the gravitational force moment these are of the form (2.2) and (2.5). The unknowns ζ, $\bar{\zeta}$, η, and $\bar{\eta}$ in this case can be determined from system (2.2), (2.6) and the analogous system for conjugate quantities with periodic and complex coefficients, since in accordance with (2.4) the functions μ_j are periodic and vary with the frequency of revolution of the craft along its orbit.

We can show that the system with periodic coefficients under consideration does not have a stable periodic solution. In fact, for system (2.2), (2.5) and the associated method of successive approximations we can construct a periodic solution in the form of a series in powers of a small parameter

$$\mu_0 = 3\omega_*^2(1-\varepsilon) \tag{2.12}$$

which for $\mu_0 = 0$ becomes some constant solution associated with the zero root of the fundamental equation of system (2.2). Omitting the actual computations, let us cite some results for the case $n_0 = 0$. The complete solution of system (2.2), (2.5) can be represented as the sum of some forced periodic solution whose stability is being investigated and of an arbitrary solution, i.e. as

$$\begin{aligned}\zeta &= D_0 + \mu_0\left[D_0^{(1)} + \frac{1}{u^{\cdot}}(D_2\sin 2u^{\cdot}t - \varepsilon_2\cos 2u^{\cdot}t)\right] + y_1e^{\mu_0 a_i(\mu_0)t}\\ \frac{d\zeta}{dt} &= \mu_0(D_2\cos 2u^{\cdot}t + \varepsilon_2\sin 2u^{\cdot}t) + y_2e^{\mu_0 a_i(\mu_0)t} \qquad \left(D_0 = -\frac{ic_3(c_1+ic_2)}{1-2c_3^2}\right)\\ \eta &= \mu_0(D_3\cos 2u^{\cdot}t + \varepsilon_3\sin 2u^{\cdot}t) + y_3e^{\mu_0 a_i(\mu_0)t}\end{aligned} \tag{2.13}$$

Here D_0 is the solution which is constant for periodic motion and which the motion under investigation becomes for $\mu_0 = 0$. The expressions for the constants $D_0^{(1)}$, D_j, and ε_j, which are completely determined, will not be given here; $a_j(\mu_0)$ are the characteristic indices of the solutions corresponding to the critical roots of the system of equations in variations for system (2.2), (2.5) in the new variables y_j.

Construction of the periodic solutions y_j in series form for the equations in variations also enables us to determine [3] the approximate values of the characteristic indices $a_j(\mu_0)$ in the form of series in powers of μ_0. The expressions for the characteristic indices can be reduced to the form

$$a_i(\mu_0) \approx a_1 + \mu_0[-(1-c_3^2)(1-3c_3^2)m_* + a_1n_*] + \ldots$$

$$\left(a_1 = \pm \frac{i}{2} c_3 \sqrt{2c_3^2 - 1} \frac{A}{H(1+\varepsilon\omega_0)}\right) \tag{2.14}$$

Here n_* is some real number and $m > 0$.

From (2.14) we find that the periodic solution under investigation is bounded in the first approximation only if the inequality

$$2C_3^2 - 1 > 0 \tag{2.15}$$

is fulfilled.

The stability of this motion is determined by the following approximation of the characteristic index. However, when inequality (2.15) is fulfilled we have

$$\mathrm{Re}\,[\mu_0 a_i(\mu_0)] = -\mu_0^2 (1 - c_3^2)(1 - 3c_3^2)\, m_* > 0$$

i.e. the periodic motion under investigation is unstable and the oriented axis deviates from the solar direction at a rate proportional to the square of the small parameter μ_0. In the general case the deviation of the oriented axis is determined by the solution of inhomogeneous Eqs. (2.2) and (2.5) which increases with time.

Unfortunately, investigations of the stability of periodic motions with allowance for perturbing moments of other types (aerodynamic, magnetic, etc.) dependent on the orientation of the craft are practically impossible without additional limiting assumptions concerning the craft design.

3. Motion about the oriented axis. In solving the above problems we assumed that $d\varphi/dt = \mathrm{const}$. If the spacecraft is in some way asymmetrical, then the angular velocity of its rotation about the oriented axis varies in accordance with the equation

$$C\frac{d^2\varphi}{dt^2} = m_z \sum_{k=1}^{N} (A_k \cos k\varphi + B_k \sin k\varphi) \tag{3.1}$$

where A_k and B_k are periodic functions of the angle of the latitude argument of the craft position; they are defined by Formula (2.8).

It is interesting to investigate the effect of a small external moment m_z on the variation of the angular velocity $d\varphi/dt$ under the assumption that the quantity $d\varphi/dt$ is of the same order as the angular velocity $u_0^{\cdot}$ of revolution of the spacecraft along its orbit.

Eq. (3.1) becomes

$$\frac{d^2\varphi}{dt^2} = \mu_0 u_0^{\cdot 2} \sum_{k=1}^{N} (A_k \cos k\varphi + B_k \sin k\varphi) \qquad \left(\mu_0 = \frac{m_z}{C u_0^{\cdot 2}}\right) \tag{3.2}$$

where μ_0 is a small dimensionless parameter.

We shall attempt to find the solution of Eq. (3.2) in the form of a series in powers of the small parameter μ_0,

$$\varphi = \varphi^{(0)} + \mu_0\varphi^{(1)} + \mu_0^2\varphi^{(2)} + \ldots \tag{3.3}$$

For the successive approximations we have Eq.

$$\frac{d^2\varphi^{(0)}}{dt^2} = 0, \qquad \frac{d^2\varphi^{(1)}}{dt^2} = u_0^{\cdot 2} I_0, \qquad \frac{d^2\varphi^{(2)}}{dt^2} = u_0^{\cdot 2} I_1 \varphi^{(1)} \tag{3.4}$$

$$\frac{d^2\varphi^{(3)}}{dt^2} = u_0^{\cdot 2}\left[I_1\varphi^{(2)} - \frac{1}{2} I_2 (\varphi^{(1)})^2\right], \ldots$$

$$\ldots, \frac{d^2\varphi^{(m)}}{dt^2} = u_0^{\cdot 2}\left[I_1\varphi^{(m-1)} + F(\varphi^{(m-2)}, \varphi^{(m-3)}, \ldots, \varphi^{(1)})\right]$$

Here

$$I_{2p} = \sum_{k=1}^{N} k^{2p} (A_k \cos k\varphi^{(0)} + B_k \sin k\varphi^{(0)}) \tag{3.5}$$

$$I_{2p+1} = \sum_{k=1}^{N} k^{2p+1}(-A_k \sin k\varphi^{(0)} + B_k \cos k\varphi^{(0)})$$

In these functions $\varphi^{(0)}$ is the zeroth approximation of solution (3.3), i.e. the solution of Eq. (3.2) for $\mu_0 = 0$,

$$\varphi^{(0)} = \varkappa t + \chi \qquad (\varkappa,\ \chi = \text{const}) \tag{3.6}$$

The motion for which the quantity $d\varphi/dt$ remains bounded corresponds to the periodic solution of Eq. (3.2). We therefore pose the problem of determining and investigating the stability of periodic revolutions (of period T) of the revolutions of the satellite along the orbit of the solutions of Eq. (3.2).

The condition of periodicity of the first-order approximation $d\varphi^{(1)}/dt$, i.e. the condition

$$\int_0^T I_0\, dt = 0 \tag{3.7}$$

is fulfilled if $\varkappa$ is a multiple of the frequency of revolution of the satellite along its orbit, i.e. if

$$\varkappa = nu_0\,, \qquad n = 0,\ \pm 1,\ \ldots \tag{3.8}$$

Three types of periodic motion are possible in this case:

$$\begin{aligned}
&1)\ n = 0, \quad (a_{0k} \cos k\chi + b_{0k} \sin k\chi) + \ldots + (a_{0N} \cos N\chi + b_{0N} \sin N\chi) = 0 \\
&2)\ n = +\nu/k, \quad (a_{\nu k} + d_{\nu k}) \cos (\nu u_0 - k\chi) + (c_{\nu k} - b_{\nu k}) \sin (\nu u_0 - k\chi) = 0 \\
&\ n = -\nu/k, \quad (a_{\nu k} - d_{\nu k}) \cos (\nu u_0 + k\chi) + (c_{\nu k} + b_{\nu k}) \sin (\nu u_0 + k\chi) = 0 \\
&3)\ n \neq 0, \quad n \neq \pm \nu/k
\end{aligned} \tag{3.9}$$

The first case corresponds to zero angular velocity for $\mu_0 = 0$ and requires special attention. The second type is associated with a finite number of solutions, since the numbers ν and k are bounded. By the same token the third case subsumes an infinite set of solutions.

Denoting the result of double integration of the functions I_{2p} and I_{2p+1} for periodic motion to within the constant factor $(u_0{}^{\cdot})^{-2}$ by $I_{2p}{}^*$ and $I_{2p+1}{}^*$, we can write the expression

$$\varphi^{(1)} = -I_0{}^* + C_1{}^{(1)}t + C_2{}^{(1)} \tag{3.10}$$

where $C_1^{(1)}$ and $C_2^{(1)}$ are integration constants.

The condition of periodicity of the second-order approximation $d\varphi^{(2)}/dt$ implies that

$$H_1 T + C_1{}^{(1)} J_2(T) + C_2{}^{(1)} J_1(T) = 0 \qquad \left(J_1(T) = \int_0^T I_1\, dt,\ J_2(T) = \int_0^T I_1 t\, dt\right) \tag{3.11}$$

where H_1 is a constant which is the product of the two periodic functions $I_0{}^*$ and I_1. If the periodic function I_1 has a constant component I_{10}, then the solution of Eq. (3.11) can be written as

$$C_1{}^{(1)} = 0, \qquad C_2{}^{(1)} = H_1 / I_{10} \tag{3.12}$$

On the other hand, if I_1 does not contain a constant component, then

$$C_1{}^{(1)} = H_1 T / J_2(T) \tag{3.13}$$

while the value of $C_2^{(1)}$ remains undetermined. Solution (3.12) is theoretically possible only for the first two cases of (3.9) and enables us to construct a solution in which $d\varphi^{(2)}/dt$ and $\varphi^{(2)}$ are periodic functions. Solution (3.13) corresponds to the third case, and, as we shall show below, in certain cases to the first case of (3.9); the solution $\varphi^{(2)}$ is an aperiodic function.

Making use of the above procedure for constructing higher-order approximations, we can obtain the solution of initial Eq. (3.2) either in the form

$$\frac{d\varphi}{dt} = \varkappa + \sum_{m=1}^{\infty} (-1)^m \mu_0{}^m \frac{d\Pi_m}{dt} \tag{3.14}$$

$$\varphi = \varkappa t + \varkappa + \sum_{m=1}^{\infty} (-1)^m \mu_0^m \left(\Pi_m - \frac{H_m}{I_{10}}\right) \quad \text{for} \quad I_{10} \neq 0$$

or in the form

$$\frac{d\varphi}{dt} = \varkappa + \sum_{m=1}^{\infty} \mu_0^m \left[\frac{d\Pi_m'}{dt} + \frac{H_m'}{J_2(T)}\right]$$

$$\varphi = \varkappa t + \chi + \sum_{m=1}^{\infty} \mu_0^m \left[\Pi_m' + \frac{H_m}{J_2(T)} t + C_2^{(m)}\right] \quad \text{for} \quad I_{10} = 0 \tag{3.15}$$

where Π_m and Π_m' are periodic functions which include several sine and cosine functions of frequencies which are multiples of the frequency u_0 , the quantities H_m and H_m' are constants.

If $\varphi = \varphi^*$ is a solution of either of the forms (3.14), (3.15), then the equation in variations for initial Eq. (3.2) becomes

$$\frac{d^2 z}{dt^2} = \mu_0 u_0^{\cdot 2} I_1 (\varphi^*) z \tag{3.16}$$

where for solution (3.14) the function

$$I_1 (\varphi^*) = I_1 (\varphi^{(0)}) - \mu_0 \varphi^{(1)} I_2 (\varphi^{(0)}) + \ldots \tag{3.17}$$

is a periodic function of time, while for solution (3.15) the quantity $I_1(\varphi^*)$ is a bounded and continuous, but aperiodic, function of time.

The method for investigating the stability of periodic motions of the (3.14) type, when equation in variations (3.16) has periodic coefficients, is described with sufficient detail in [3]. Carrying out all the required operations, we can establish that in the case under consideration the equation in variations has a bounded solution only for

$$I_{10} < 0 \tag{3.18}$$

If $I_1(\varphi^*)$ is an aperiodic function of time, then equation in variations (3.16) can be written as a system of two Eqs.

$$dz_1/dt = z_2, \qquad dz_2/dt = \mu_0 u_0^{\cdot 2} I_1 (\varphi^*) z_1 \tag{3.19}$$

Considering (3.19) as a special form of a system of linear differential equations with variable coefficients the stability of whose solutions is investigated, for example, in [4], we can readily establish that system (3.19) has two eigenvalues whose sum is zero. Hence, the only possible case in which system (3.19) can have a bounded solution is when each of its eigenvalues is equal to zero. The necessary condition for this is that

$$I_1 (\varphi^*) \to 0 \qquad \text{as} \qquad t \to \infty \tag{3.20}$$

since fulfillment of this condition means that system (3.19) has eigenvalues equal to the eigenvalues of the system with constant coefficients

$$dz_1 / dt = z_2, \qquad dz_2 / dt = 0 \tag{3.21}$$

which results from the initial system upon elimination of the terms with variable coefficients. System (3.21) has two eigenvalues equal to zero. Condition (3.20) is not fulfilled, so that system (3.19) has an eigenvalue smaller than zero, and motion (3.15) under investigation is unstable.

Thus, Eq. (3.2) has a periodic solution (3.14) for $I_{10} \neq 0$; the necessary condition for the stability of the solution is inequality (3.18),

$$I_{10} = \sum_{k=1}^{N} k (- a_{0k} \sin k\varkappa + b_{0k} \cos k\varkappa) \qquad \text{for} \quad n = 0 \tag{3.22}$$

where χ is the solution of Eq.

$$I_{00} = \sum_{k=1}^{N} (a_{0k} \cos k\chi + b_{0k} \sin k\chi) = 0 \tag{3.23}$$

Writing the expressions for I_{10} and I_{00} as

$$I_{10} = \sum_{k=1}^{N} k\Phi_k \cos(k\chi + \psi_k), \qquad I_{00} = \sum_{k=1}^{N} \Phi_k \sin(k\chi + \psi_k)$$
$$(\Phi_k = \sqrt{a_{0k}^2 + b_{0k}^2}, \quad a_{0k} = \Phi_k \sin\psi_k, \quad b_{0k} = \Phi_k \cos\psi_k) \tag{3.24}$$

and rejecting the solution $\chi = -\psi$, $\chi = -\psi + \pi$ of eq. (3.23) (this solution corresponds to the limiting condition $\psi_k = k\psi$), we can establish that the quantity $|I_{10}| = \sqrt{I_{10}^2}$ is a fixed sign quadratic form of the quantities Φ_k, and hence of a_{0k} and b_{0k} for values of χ equal to the roots of Eq. (3.23) for $k = 1, 2$ only.

Thus, only when the expression for the external force moment M_z includes just two harmonics in the argument φ can we draw a reliable conclusion concerning the stability or instability of the constructed periodic motion. We note that the expansion

$$(A_1 \cos\varphi + B_2 \sin\varphi) + (A_2 \cos 2\varphi + B_3 \sin 2\upsilon)$$

includes practically interesting cases of the gravitational force moment and the magnetic force moment computed under the assumption that the Earth's magnetic field can be approximated by a dipole whose axis coincides with that of the Earth.

If the series I_{00} and I_{10} contain only the k-th harmonic, then Eq. (3.23) has the two solutions $k\chi + \psi_k = 0$ and $k\chi + \psi_k = \pi$, where $I_{10} = -kA_k < 0$ for $k\chi + \psi_k = \pi$.

For $n = \pm\nu/k$ the constants $I_{0\nu k}$ and $I_{1\nu k}$ are also reducible to the form (3.24), which in this case contains just one harmonic; hence, the necessary stability condition (3.18) is always fulfilled for periodic motions corresponding to the second case of (3.9).

In the general case of a aperiodic solution (3.3) of Eq. (3.2) in which the value of $\varkappa$ in (3.6) is not a multiple of the quantity $u_0{}^{\boldsymbol{\cdot}}$ we have a linear variation in time of the angular velocity of natural rotation of the satellite which is proportional to the second power of the small parameter μ_0. With this kind of variation the quantity $d\varphi/dt$ very likely attains a level $d\varphi/dt = nu_0{}^{\boldsymbol{\cdot}}$ for which the required conditions for stability of the periodic motion are fulfilled.

Thus, the steady motion of a spacecraft about the oriented axis under the action of a small external force moment is always a periodic state in which the quantity $d\varphi/dt$ varies periodically and with a small amplitude about some level which is a multiple of the frequency of revolution of the satellite along its orbit. This justifies to some extent the assumption that the quantity $d\varphi/dt$ is constant which we made in investigating the motion of the craft axis.

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